

QUANTUM GEOMETRY OF FIELD EXTENSIONS

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ABSTRACT We show that noncommutative differential forms on $k[x]$, k a field, are of the form $\Omega^1 = k_\lambda[x]$ where $k_\lambda \supseteq k$ is a field extension. We compute the case $\mathbb{C} \supset \mathbb{R}$ explicitly, where Ω^1 is 2-dimensional. We study the induced quantum de Rahm complex, its cohomology and the associated moduli space of flat connections.

1 INTRODUCTION

Let A be an algebra, which we consider as playing the role of ‘co-ordinates’ in algebraic geometry, except that we do not require the algebra to be commutative. The appropriate notion of cotangent space or differential 1-forms in this case is [1]

1. Ω^1 an A -bimodule
2. $d : A \rightarrow \Omega^1$ a linear map obeying the Leibniz rule $d(ab) = adb + (da)b$ for all $a, b \in A$.
3. The map $A \otimes A \rightarrow \Omega^1$, $a \otimes b \mapsto adb$ is surjective.

When A has a Hopf algebra structure with coproduct $\Delta : A \rightarrow A \otimes A$ and counit $\epsilon : A \rightarrow k$ (k the ground field), we say that Ω^1 is *bicovariant* if

4. Ω^1 is a bicomodule with coactions $\Delta_L : \Omega^1 \rightarrow A \otimes \Omega^1$, $\Delta_R : \Omega^1 \rightarrow \Omega^1 \otimes A$ bimodule maps (with the tensor product bimodule structure on the target spaces, where A is a bimodule by left and right multiplication).

5. d is a bicomodule map with the left and right regular coactions on A provided by Δ .

A morphism of calculi means a bimodule and bicomodule map forming a commuting triangle with the respective d maps. One says [2] that a calculus is *coirreducible* if it has no proper quotients.

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The main difference is that, in usual algebraic geometry, the multiplication of forms Ω^1 by ‘functions’ A is the same from the left or from the right. However, if $adb = (db)a$ then by axiom 2. we have $d(ab-ba) = 0$, i.e. we cannot naturally suppose this when A is noncommutative. This possible noncommutativity of forms and ‘functions’ is the main generalisation featuring in the above axioms. We say that a differential calculus is noncommutative or ‘quantum’ if the left and right multiplication of forms by functions do not coincide. It turns out that many geometrical constructions assume neither commutativity of A nor commutativity of the differential calculus. See [3] for a theory of bundles with quantum group fiber and the example of the q -monopole over a q -deformed sphere. Moreover, there are natural prolongations to higher order differential forms, i.e. the entire exterior algebra Ω^* once Ω^1 is specified. Note that our approach is somewhat different from [4], where an entire Ω^* on an algebra is effectively specified via a ‘spectral triple’.

Bicovariant quantum differential calculi on strict quantum groups and group algebras over \mathbb{C} have recently been classified in [2]. This class of structures is, however, interesting even when A is commutative, being a generalisation of usual concepts of differential forms. In this paper we study the simplest case, where $A = k[x]$, the polynomials over a field k , with its additive coproduct and counit

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon x = 0.$$

A complete classification of the bicovariant Ω^1 in this case is provided in Section 2. They turn out to be of the form $\Omega^1 = k_\lambda[x]$ where $k_\lambda \supset k$ is a field extension. In Section 3 we study a concrete example on $\mathbb{R}[x]$ associated to the field extension $\mathbb{C} \supset \mathbb{R}$. Section 4 studies the exterior algebra, de Rahm cohomology and elements of gauge theory in this setting. Section 5 concludes with some further quantum geometric considerations.

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2 Ω^1 and field extensions

When Ω^1 is required to be bicovariant, there is a standard argument[1] that it must be of the form

$$\Omega^1 = \Omega_0 \otimes A, \quad da = (\pi \otimes \text{id})(\Delta a - 1 \otimes a)$$

where $\Omega_0 = \ker \epsilon / \mathcal{M}$ with canonical projection $\pi : \ker \epsilon \rightarrow \Omega_0$ and \mathcal{M} is a left ideal contained in $\ker \epsilon$ and stable under the Hopf algebra adjoint coaction Ad . The right (co)module structures are those of A alone by (co)multiplication. The left (co)module structures are the tensor product of those on Ω_0 as inherited from $\ker \epsilon \subset A$ (where A acts by left multiplication and coacts by Ad) and those on A by (co)multiplication. We recall that modules and comodules of a Hopf algebra A have a tensor product induced by the coproduct and product of A respectively. Then bicovariant Ω^1 are in 1-1 correspondence with the Ad -stable left ideals $\mathcal{M} \subset \ker \epsilon$. When A is cocommutative the adjoint coaction Ad is trivial.

Proposition 2.1 *When $A = k[x]$, the coirreducible bicovariant Ω^1 are in 1-1 correspondence with irreducible monic polynomials $m \in k[x]$, and take the form $\Omega^1 = k_\lambda[x]$ where $k_\lambda = k[\lambda]/\langle m \rangle$ is the corresponding field extension. The bimodule structures and differential are*

$$f(x) \cdot P(\lambda, x) = f(x + \lambda)P(\lambda, x), \quad P(\lambda, x) \cdot f(x) = P(\lambda, x)f(x), \quad df(x) = \frac{f(x + \lambda) - f(x)}{\lambda}$$

for all $f \in k[x], P \in k_\lambda[x]$.

Proof According to the above, bicovariant differential calculi on $k[x]$ are in 1-1 correspondence with ideals $\mathcal{M} \subset \ker \epsilon$. Here $\ker \epsilon = \langle x \rangle$, the ideal generated by x in $k[x]$. Since $k[x]$ is a P.I.D., the ideal \mathcal{M} above is generated by a polynomial. Since $\mathcal{M} \subset \ker \epsilon$, this polynomial is divisible by x , i.e. $\mathcal{M} = \langle xm \rangle$. Coirreducible calculi correspond to m irreducible and monic.

We identify the corresponding $\Omega_0 = \langle x \rangle / \langle xm \rangle \cong k[\lambda] / \langle m \rangle = k_\lambda$ by $xf(x) \mapsto f(\lambda)$. Under this identification, $\Omega^1 = \Omega_0 \otimes k[x] \cong k_\lambda[x]$. The action from the right is by the inclusion $k[x] \subset k_\lambda[x]$. The action from the left is by

$$f(x) \cdot x^m \otimes x^n = f(x \otimes 1 + 1 \otimes x)x^m \otimes x^n.$$

as the tensor product action. Hence $f(x) \cdot \lambda^{m-1}x^n = f(\lambda + x)\lambda^{m-1}x^n$ under our identification. The quotient by $\langle xm(x) \rangle$ or $\langle m(\lambda) \rangle$ is understood in these expressions.

We compute $df = f(x \otimes 1 + 1 \otimes x) - 1 \otimes f(x)$ modulo $\langle xm \rangle$ in the first tensor factor. Under our isomorphism this is $f(\lambda + x) - f(x)/\lambda$ modulo $\langle m(\lambda) \rangle$. Note that $dx = x \otimes 1$ modulo $\langle xm \rangle$ become $dx = 1 \in k_\lambda[x]$.

To see explicitly that the correspondence here is indeed 1-1, suppose that $k_{\lambda_1}[x] \cong k_{\lambda_2}[x]$ as quantum differential calculi associated to $m_1(\lambda_1)$ and $m_2(\lambda_2)$. Since the isomorphism is in particular a right module map under $k[x]$, it restricts to the identity on $k[x]$. And since the isomorphism forms a commutative triangle with the d maps, it identifies $\frac{f(x+\lambda_1)-f(x)}{\lambda_1}$ with $\frac{f(x+\lambda_2)-f(x)}{\lambda_2}$ for all f . Taking $f = x^2$ we have $2x + \lambda_1$ and $2x + \lambda_2$ identified, hence λ_1, λ_2 identified. Similarly by induction λ_1^n and λ_2^n are identified for all $n \geq 0$. One can also use the left module map property to conclude this. Hence $m_1(\lambda_2) = 0$ in k_{λ_2} . Hence m_2 divides m_1 . Since m_1 is monic and irreducible, we conclude that $m_1 = m_2$ as required. The converse direction is clear. \square

This generalises the observation in [5] that coirreducible bicovariant quantum differential calculi over $\mathbb{C}[x]$ are parametrized by $\lambda_0 \in \mathbb{C}$ (say). Here $m(\lambda) = \lambda - \lambda_0$ and $\pi(\lambda) = \lambda_0$. Hence, in this case,

$$df = dx \frac{f(x + \lambda_0) - f(x)}{\lambda_0}. \quad (1)$$

The ratio on the right should be understood as the coefficient of λ_0 in $f(x + \lambda_0) - f(x)$, i.e. we include the usual differential calculus as the case $\lambda_0 = 0$. More generally, if the extension is Galois, the roots λ_i of m are as many as its degree and are primitive elements of k_λ , i.e. $k_\lambda \cong k[\lambda_i]$ by setting $\lambda = \lambda_i$, for each i . This gives us different ways of thinking of the differentials in Proposition 2.1 concretely as finite differences, all of them equivalent via the action of the Galois group of k_λ automorphisms that permute the λ_i .

It is easy to verify that Ω^1 in Proposition 2.1 is bicovariant under the left and right coactions

$$\Delta_R P(\lambda, x) = P(\lambda, x + y) \in k_\lambda[x] \otimes k[y], \quad \Delta_L P(\lambda, x) = P(\lambda, y + x) \in k[y] \otimes k_\lambda[x] \quad (2)$$

induced by the coproduct Δ , as it must be by construction. Here the coacting copy of A is denoted by $k[y]$. The space Ω_0 is the subspace of Ω^1 invariant under the left coaction Δ_L ,

again by the general theory. Clearly, the dimension of Ω_0 over k , which is the dimension of the quantum differential calculus, is the degree of m , the degree of the associated field extension. The elements $\{1 = dx, \lambda, \dots, \lambda^{\deg(m)-1}\}$ of $k_\lambda[x]$ are a basis of right-invariant 1-forms.

3 Quantum differentials for the complex extension of the reals

In this section we consider in detail the case $k = \mathbb{R}$ and $m(\lambda) = \lambda^2 + 1$. Then $k_\lambda = \mathbb{C}$. The space of right-invariant 1-forms has basis

$$\Omega_0 = \{1 = dx, \quad \lambda = dx^2 - 2(dx)x\}$$

We use the notations dx and $\omega \equiv dx^2 - 2(dx)x = xdx - (dx)x$ (by the Leibniz rule) for these two 1-forms in what follows.

Lemma 3.1 *The left part of the bimodule structure on Ω^1 in this basis is given by*

$$x \cdot dx = (dx)x + \omega, \quad x \cdot \omega = \omega x - dx$$

Proof The first equality is the definition of ω (given the Leibniz rule). The second depends on the irreducible polynomial m according to $x \cdot \omega = (x + \lambda)\lambda = \lambda x - 1 = \omega x - dx$. \square

Proposition 3.2 *The exterior differential is given by*

$$df(x) = (dx)\Im f(x+i) + \omega(f(x) - \Re f(x+i))$$

where $f \in \mathbb{R}[x]$ is continued to \mathbb{C} and \Im, \Re denote imaginary and real parts. The left and right multiplication of forms by functions are related by

$$f(x) \cdot \begin{pmatrix} dx & \omega \end{pmatrix} = \begin{pmatrix} dx & \omega \end{pmatrix} \begin{pmatrix} \Re & -\Im \\ \Im & \Re \end{pmatrix} f(x+i)$$

Proof This follows directly as an example of Proposition 2.1 on writing $\lambda = i$. Here we provide a more conventional direct proof based on the more conventional description in Lemma 3.1. First we write Lemma 3.1 in matrix form

$$x \cdot \begin{pmatrix} dx & \omega \end{pmatrix} = \begin{pmatrix} dx & \omega \end{pmatrix} \begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$$

Then $f(x) \cdot dx = (dx)f(x + \Lambda)^1_1 + \omega f(x + \Lambda)^2_1$ where $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the numerical indices denote the matrix element. We regard x for these purposes as multiplied by the identity matrix. Similarly for $f(x) \cdot \omega$.

Now, by induction on the Leibniz rule,

$$\begin{aligned} dx^m &= x^{m-1}dx + x^{m-2}(dx)x + \cdots + (dx)x^{m-1} \\ &= (dx \quad \omega) (x^{m-1} + x^{m-2}(x + \Lambda) + \cdots + x(x + \Lambda)^{m-2} + (x + \Lambda)^{m-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (dx \quad \omega) \left(\frac{(x + \Lambda)^m - x^m}{\Lambda} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (dx \quad \omega) ((x + \Lambda)^m - x^m) \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned}$$

This provides the formula

$$df = (dx \quad \omega) (f(x + \Lambda) - f(x)) \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Since $\Lambda^2 = -1$, we then identify the Λ^0 and Λ^1 parts of $f(x + \Lambda) - f(x)$ with the real and imaginary parts of $f(x + i) - f(x)$ as stated. Similarly for $f(x) \cdot dx$ and $f(x) \cdot \omega$. \square

To gain further insight into this differential calculus it is useful to embed it in the 1-parameter family corresponding to $m(\lambda) = \lambda^2 + q^2$, $q \in \mathbb{R}$. This is isomorphic to the above $q = 1$ case for all $q \neq 0$ and not irreducible for $q = 0$. It does, however, have an interesting limit as $q \rightarrow 0$. Briefly, the relevant formulae are

$$x \cdot (dx \quad \omega) = (dx \quad \omega) \begin{pmatrix} x & -q^2 \\ 1 & x \end{pmatrix}, \quad (3)$$

resulting in

$$df = q^{-2}\omega(f(x) - \Re f(x + iq)) + q^{-1}(dx)\Im f(x + iq). \quad (4)$$

This has a limit as $q \rightarrow 0$:

$$x\omega = \omega x, \quad xdx = (dx)x + \omega, \quad df = \omega \frac{1}{2}f'' + (dx)f' \quad (5)$$

in terms of the usual newtonian derivative f' . This is the 2-jet calculus in [5] whereby up to second order derivatives are viewed as ‘first order’ with respect to the new calculus and an appropriate ‘braided derivation’[2] rule. We see that this calculus, although not coirreducible, arises naturally as a degenerate limit of coirreducibles corresponding to the extension $\mathbb{R} \subset \mathbb{C}$.

4 Quantum cohomology and gauge theory of field extensions

In this section, we consider two natural prolongations of the $\Omega^1(k[x])$ associated to a field extension to ‘exterior algebras’ $\Omega^n(k[x])$ of degree $n > 1$. We compute the first quantum cohomology for each prolongation in the case of the extension $\mathbb{R} \subseteq \mathbb{C}$, and the associated gauge theory.

We recall first that a differential graded algebra Ω^\bullet over a unital algebra A means a graded algebra with degree zero part A itself, and $d : \Omega^\bullet \rightarrow \Omega^\bullet$ which increases the degree by 1 and obeys $d^2 = 0$ and the graded Leibniz rule. In other words, Ω^\bullet has the algebraic properties of an ‘exterior algebra’ in DeRahm theory and one may likewise compute its ‘quantum de Rahm cohomology’. Thus,

$$H^1 = \{\omega \in \Omega^1 \mid d\omega = 0\} / \{da \mid a \in A\}. \quad (6)$$

Given Ω^1 , its maximal prolongation is defined as follows. First of all, we recall that view of Axiom 3 above, we can write Ω^1 as a quotient of the universal calculus $\Omega_U^1 = \ker(\cdot : A \otimes A \rightarrow A)$ by a sub-bimodule \mathcal{N} . Here Ω_U^1 has the obvious bimodule structure from $A \otimes A$ and $d_U a = a \otimes 1 - 1 \otimes a$. (Note that when A is a Hopf algebra then $A \otimes A \cong A \otimes A$ by $a \otimes b \mapsto (\Delta a)b$ restricts to $\Omega_U^1 \cong \ker \epsilon \otimes A$ and $\mathcal{N} \cong \mathcal{M} \otimes A$ giving the description used in Section 2). Moreover, Ω_U^1 is the degree 1 part of a canonical Ω_U (albeit with trivial quantum cohomology). Here $\Omega_U^n \subset A^{\otimes n+1}$ as elements in the joint kernel of all product maps \cdot_i multiplying the $i, i+1$ ’th copies of A . This can also be identified with $\Omega_U^n = \Omega_U^1 \otimes_A \cdots \otimes_A \Omega_U^1$ in the obvious way; see [4][3]. Here

$$d_U(a_0 \otimes a_1 \cdots \otimes a_n) = \sum_{i=0}^{n+1} (-1)^{n+1-i} a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \quad (7)$$

One may check that $d_U \circ d_U = 0$. The product \wedge of Ω_U is given by multiplication between the two adjacent copies of A . A general Ω^\bullet over A is a quotient of Ω_U by a differential graded ideal (i.e. an ideal stable under d_U). Without loss of generality we assume that the degree 0 part of the ideal is zero. The degree 1 part is some subbimodule $\mathcal{N} \subseteq \Omega_U^1$ and conversely, given \mathcal{N} the maximal prolongation is provided by the differential ideal generated by \mathcal{N} . Its degree 2 part is $\mathcal{F} = \Omega_U^1 \wedge \mathcal{N} + \mathcal{N} \wedge \Omega_U^1 + d_U \mathcal{N}$ and $\Omega^2 = \Omega_U^2 / \mathcal{F}$.

Lemma 4.1 *For the field extension $\mathbb{R} \subset \mathbb{C}$, the maximal Ω^2 is generated as an $\mathbb{R}[x]$ -module by the two forms $dx \wedge dx$ and $dx \wedge \omega$. Moreover,*

$$d\omega = 2dx \wedge dx = 2\omega \wedge \omega, \quad dx \wedge \omega = -\omega \wedge dx.$$

Proof The subbimodule \mathcal{N} in our case is generated by $x\omega - \omega x + dx$ where ω is defined as above. Now, $d\omega = d(xdx - (dx)x) = 2dx \wedge dx$ from the definition of ω and the graded Leibniz rule and $d^2 = 0$. Hence the subbimodule \mathcal{F} is generated by $\Omega_U^1 \wedge \mathcal{N}, \mathcal{N} \wedge \Omega_U^1$ and $dx \wedge \omega + \omega \wedge dx + 2xdx \wedge dx - 2(dx \wedge dx)x$. From Lemma 3.1 we have $xdx \wedge dx = (dx) \wedge xdx + \omega \wedge dx = (dx \wedge dx)x + dx \wedge \omega + \omega \wedge dx$ up to terms in $\Omega_U^1 \wedge \mathcal{N}, \mathcal{N} \wedge \Omega_U^1$. Therefore, \mathcal{F} is generated by these and $\omega \wedge dx + dx \wedge \omega$.

Finally, from the definition of ω , the relations in Ω^1 and \mathcal{N} , we have

$$\begin{aligned} \omega \wedge \omega &= (xdx - (dx)x) \wedge \omega = -x\omega \wedge dx - (dx)x \wedge \omega \\ &= dx \wedge dx - \omega \wedge xdx - (dx)x \wedge \omega \\ &= dx \wedge dx - \omega \wedge \omega - \omega \wedge dxx + dx \wedge dx - (dx) \wedge \omega x = 2dx \wedge dx - \omega \wedge \omega \end{aligned}$$

which gives the stated description of Ω^2 as a quotient of the tensor square over $\mathbb{R}[x]$ of Ω^1 . \square

Proposition 4.2 *With the maximal Ω^2 , the quantum de Rham cohomology H^1 associated to $\mathbb{R} \subset \mathbb{C}$ vanishes.*

Proof Suppose $d((dx)f + \omega g) = 0$ i.e. $-dx \wedge df + 2(dx \wedge dx)g - \omega \wedge dg = 0$. Put in the form of df and dg from Proposition 3.2 and we see this is equivalent to

$$\Im g(x+i) = f(x) - \Re f(x+i), \quad \Re g(x+i) + g(x) = \Im f(x+i)$$

which can be combined into the single equation

$$f(x+i) - f(x) = i(g(x) + g(x+i)). \tag{8}$$

We now show that such f, g are necessarily of the form

$$f = \Im h(x+i), \quad g = h(x) - \Re h(x+i)$$

for some $h(x)$. Note first that if (f, g) obey (8) and without loss of generality $f = nx^{n-1} + \text{lower degree}$, say, then

$$g = \frac{n(n-1)}{2}x^{n-2} + \text{lower degree}.$$

Indeed, writing $g = \mu x^p + \text{lower degree}$, the second half of (8) implies that $\mu x^p + \mu \Re(x + \iota)^p + \dots = 2\mu x^p + \dots = n\Im(x + \iota)^{n-1} - nx^{n-1} + \dots = n(n-1)x^{n-2} + \dots$. Equating leading terms gives $p = n-2$ and $2\mu = n(n-1)$.

Now let

$$f_n = \Im(x + \iota)^n, \quad g_n = x^n - \Re(x + \iota)^n = x^n - (x + \iota)^n + \iota f_n$$

for $n > 0$. Note that the leading term of f_n is nx^{n-1} and the leading term of g_n is $\frac{n(n-1)}{2}x^{n-2}$.

Hence

$$f = f_n + \bar{f}, \quad g = g_n + \bar{g}$$

defines two polynomials \bar{f}, \bar{g} of lower degree. Now since (f_n, g_n) are the components of the differential of x^n , and since $d^2 = 0$, we know that they obey (8). Hence (\bar{f}, \bar{g}) obeys (8) and has lower degree.

Therefore we have a proof by induction. The case where $n = 2$ is easily seen to be true. I.e. if $f = 2x + \mu$ then (8) implies as above that $g = 1$. Then indeed $f = \Im((x + \iota)^2 + \mu(x + \iota))$ and $x^2 + \mu x - \Re((x + \iota)^2 + \mu(x + \iota)) = 1 = g$ as required. In terms of differential forms, the assertion is that if $(dx)(2x + \mu) + \omega h(x)$ is closed then $h(x) = 1$ and the form is $d(x^2 + \mu x)$. This may also be verified directly from the relations in Lemma 4.1. \square

Next we consider a natural quotient of the above prolongation which always exists when A is a Hopf algebra and Ω^1 is bicovariant. In this case $\Omega^1 = \Omega_0 \otimes A$ as explained in Section 2, and Ω is defined in such a way that the invariant differential forms ‘braided-anticommute’ where the braiding is the one associated to the quantum double of $A[1]$. Fortunately, in our case where A is commutative and cocommutative, the quantum double braiding is the trivial flip map (the usual transposition). Hence in this case we have simply $\Omega^n = \Lambda^n \Omega_0 \otimes A$, where Λ^n denotes the usual exterior algebra of the vector space Ω_0 . We call this the *skew exterior algebra*.

Proposition 4.3 *The skew Ω^2 in the case $\mathbb{R} \subset \mathbb{C}$ is 1-dimensional with basis $dx \wedge \omega$ (i.e. as in*

Lemma 3.1 with the additional relations $(dx)^2 = 0 = \omega^2$. The first quantum cohomology in this case is $H^1 = \mathbb{R}\omega$, i.e. 1-dimensional and spanned by ω .

Proof This time $d((dx)f + \omega g) = 0$ and df, dg from Proposition 3.2 implies only that

$$\Im g(x + \iota) = f(x) - \Re f(x + \iota) \quad (9)$$

as the coefficient of $dx \wedge \omega$. (The first half of (8) no applies since $dx \wedge dx = 0$.) This equation still implies that if $f = nx^{n-1} + \text{lower degree}$ and $n > 2$ then $g = \frac{n(n-1)}{2}x^{n-2} + \text{lower degree}$ as before. Indeed, if $g = \mu x^p + \dots$ then it says $\mu p x^{p-1} + \dots = n \frac{(n-1)(n-2)}{2} + \dots$. This is weaker than before because it does not fix μ when $n = 2$. We proceed as before by writing $f = f_n + \bar{f}$, $g = g_n + \bar{g}$ so that \bar{f}, \bar{g} obey (9) and have lower degree. In this way we obtain (without loss of generality by scaling f, g suitably) $f = F + 2x + \mu$ and $g = G + \tau$ where $(dx)F + \omega G = dh$ for some h . Adding $f_2 + \mu f_1$ and $g_2 = 1$ (here $g_1 = 0$) to F, G , we have $(dx)f + \omega g = (1 - \tau)\omega + dh'$ for $h' = h + x^2 + \mu x$. Hence $H^1 = \mathbb{R}\omega$. Indeed $d\omega = 0$ for this choice of Ω^2 but ω is not exact. \square

Finally, associated to any Ω over a unital algebra A one has further ‘quantum geometrical’ constructions, such as gauge theory. In its simplest form we consider a gauge field as any $\alpha \in \Omega^1$ and a gauge transform as an any invertible $\gamma \in A$. The group of gauge transforms acts on the set of α by

$$\alpha^\gamma = \gamma^{-1}\alpha\gamma + \gamma^{-1}d\gamma \quad (10)$$

The fundamental lemma of gauge theory is that the curvature

$$F(\alpha) = d\alpha + \alpha \wedge \alpha \in \Omega^2 \quad (11)$$

is covariant in the sense $F(\alpha^\gamma) = \gamma^{-1}F(\alpha)\gamma$. Moreover, one can consider sections $\psi \in A$ and a covariant derivative $\nabla\psi = d\psi + \alpha\psi \in \Omega^1$. One has an action of the group of gauge transformations by $\psi^\gamma = \gamma^{-1}\psi$ and $\nabla^\gamma\psi^\gamma = (\nabla\psi)^\gamma$. These facts require only that Ω^1, Ω^2 obey the natural axioms as part of a differential graded algebra, see[5]. Note that when Ω^1 is ‘quantum’, the nonlinearity in F does not necessarily collapse even though the ‘structure group’

here is trivial, i.e. one has many of the features of nonAbelian gauge theory. One may also consider α with values in some other algebra.

In our present setting where $A = k[x]$, only 1 will be invertible as a polynomial. One may enlarge A and our constructions above to handle this. Alternatively, instead of the ‘finite’ gauge transformations γ one can consider only ‘infinitesimal’ ones. Here an infinitesimal gauge transformation means $\theta \in k[x]$ acting by

$$\alpha^\theta = \alpha + d\theta + \alpha\theta - \theta\alpha, \quad F(\alpha^\theta) = F(\alpha) + F(\alpha)\theta - \theta F(\alpha) \quad (12)$$

to lowest order in θ . This can be stated more formally as a vector field associated to each θ on the space of connections, etc., in the usual way. The covariant derivative $\nabla = d + \alpha \wedge$ is covariant to lowest order under $\psi^\theta = \psi - \theta\psi$. By the same methods as in [5] one may check that any Ω^1, Ω^2 which are part of an exterior algebra will do for these features of gauge theory. Covariance of the curvature means that the vector fields associated to θ restrict to vector fields on the space of flat connections. They may not, however, restrict to only the algebraic (i.e. polynomial) part.

Proposition 4.4 *For the extension $\mathbb{R} \subset \mathbb{C}$ and the maximal prolongation Ω^2 we write $\alpha = (dx)a + \omega b$ and $F(\alpha) = (dx)^2 F_0 + dx \wedge \omega F_1$, say, then*

$$F_0 + \imath F_1 = (a(x) + \imath(b(x) + 1))(a(x + \imath) - \imath(b(x + \imath) + 1)) - 1$$

and the infinitesimal gauge transformations are

$$(a(x) + \imath(b(x) + 1)) \mapsto (a(x) + \imath(b(x) + 1))(1 + \theta(x) - \theta(x + \imath)).$$

The algebraic part of the space of flat connections is a circle

$$\text{Flat} = \{dxs + \omega t \mid s, t \in \mathbb{R}, \quad s^2 + (t + 1)^2 = 1\} = S^1 \subset \mathbb{C}.$$

Here $s + \imath t \in \mathbb{C} \subset \mathbb{C}[x] \subset \Omega^1$ is a circle of unit radius centered at $-\imath$. The action of $\theta(x) = x$ is a unit vector field along the circle, so that the algebraic moduli space is the class of the zero connection.

Proof We use Proposition 3.2 to compute $d\alpha + \alpha \wedge \alpha$ in Ω^2 . We then use Lemma 4.1 and collect the coefficients of $dx \wedge dx$ and $dx \wedge \omega$ as

$$F_0 = (-\Im a(x + \iota) + \Re b(x + \iota))(1 + b(x)) + b(x) + (\Re a(x + \iota) + \Im b(x + \iota))a(x)$$

$$F_1 = (\Re a(x + \iota) + \Im b(x + \iota))(1 + b(x)) - a(x) + (\Im a(x + \iota) - \Re b(x + \iota))a(x).$$

Likewise from Proposition 3.2, the action of infinitesimal gauge transformation $\theta \in \mathbb{R}[x]$ is

$$a \mapsto a(x)(1 + \theta(x)) + \Im \theta(x + \iota)(1 + b(x)) - \Re \theta(x + \iota)a(x)$$

$$b \mapsto b(x)(1 + \theta(x)) + \theta - \Re \theta(x + \iota)(1 + b(x)) - \Im \theta(x + \iota)a(x).$$

We can then combine these expressions into the expressions shown for $F_0 + \iota F_1$ and $a + \iota b$. Note that $\alpha = dxa + \omega b = a + \iota b$ in the identification of Proposition 2.1, and similarly $F(\alpha) = dx \wedge (F_0 + \iota F_1)$ by an extension of this identification.

Next we compute the algebraic part of the space of flat connections. Suppose that

$$\alpha = a + \iota b = sx^n + \dots + \iota(tx^m + \dots), \quad s, t \neq 0$$

are the leading terms for the real and imaginary parts. Here $n, m \geq 0$. Then

$$\begin{aligned} F(\alpha) &= -snx^{n-1} - \iota s \frac{n(n-1)}{2} x^{n-2} + 2tx^m + \iota tmx^{m-1} \\ &\quad + (sx^n + \iota tx^m)(sx^n - \iota tx^m + \iota snx^{n-1} + tmx^{m-1} - s \frac{n(n-1)}{2} x^{n-2} + \iota t \frac{m(m-1)}{2} x^{m-2}) + \dots \\ &= -snx^{n-1} - \iota s \frac{n(n-1)}{2} x^{n-2} + 2tx^m + \iota tmx^{m-1} + s^2 x^{2n} + t^2 x^{2m} \\ &\quad + \iota s^2 nx^{2n-1} + \iota t^2 mx^{2m-1} + st(m-n)x^{m+n-1} + \frac{\iota}{2} st(m(m-1) - n(n-1))x^{m+n-2} + \dots \end{aligned}$$

Now, since $n \geq 0$ the x^{n-1} and ιx^{n-2} terms can be dropped against the x^{2n} and ιx^{2n-1} terms respectively.

Suppose that $m \geq 1$. Then the x^m and ιx^{m-1} terms can likewise be dropped. If $m = n$ then $(s^2 + t^2)x^{2n} + \iota(s^2 + t^2)nx^{2n-1}$ is dominant, in which case $F = 0$ would imply $s = 0, t = 0$. So this case is excluded under our initial assumption. If $m > n$ then $t^2 x^{2m} + \iota t^2 mx^{2m-1}$ is dominant, in which case $t = 0$. Likewise $m < n$ would imply $s = 0$.

Hence $m = 0$ for a flat connection under our assumption $s, t \neq 0$. In this case, if $n \geq 1$ then $s^2 x^{2n} + \iota s^2 nx^{2n-1}$ is dominant and $F = 0$ would imply $s = 0$. Hence $n = 0$ as well for a flat connection.

It remains to consider the simpler cases where $t = 0$ or $s = 0$ in our leading terms (i.e. real or imaginary α). If $t = 0$ and $s \neq 0$ we similarly conclude that $n = 0$ for a non-zero flat connection. And if $s = 0$ and $t \neq 0$ then $m = 0$ for a non-zero flat connection in the same way. Hence for an algebraic connection of zero curvature, we are left with $\alpha = s + \imath t$ for $s, t \in \mathbb{R}$. Then

$$F(dxs + \omega t) = d(dxs + \omega t) + (dxs + \omega t) \wedge (dxs + \omega t) = (t^2 + 2t + s^2)dx \wedge dx$$

via Lemma 4.1, which tells us that $s^2 + (t + 1)^2 = 1$ for zero curvature.

For $\theta(x) = x\epsilon$, where $\epsilon \in \mathbb{R}$, we have the infinitesimal gauge transform $s + \imath(t + 1) \mapsto (s + \imath(t + 1))(1 - \imath\epsilon)$ to lowest order in ϵ . This is an infinitesimal rotation of $s + \imath t$ about $-\imath$. \square

Although we can consider only infinitesimal gauge transformations in our present algebraic setup, it is clear that the exponentiation of the infinitesimal gauge transformations associated to $\theta(x) = x\epsilon$ rotate us around the stated S^1 . Since this S^1 passes through the origin, we see that all the algebraic zero curvature solutions stated are connected in this way to the zero connection by finite gauge transformations. Note also that infinitesimal gauge transformations by $\theta(x) = x^n\epsilon$, $n > 1$ take us out of the space of algebraic zero curvature connections. This tells us that additional zero curvature connections beyond those in the proposition certainly exist in a suitable context, just not as polynomials. For example, the formal exponentiation of the gauge transform by $\theta(x) = x^2\epsilon$ of the $\alpha = -2\imath$ solution is

$$\alpha = -\imath(1 + e^{\tau(1-2x\imath)}), \quad \tau \in \mathbb{R}.$$

It corresponds to the gauge transformation of $\alpha = -2\imath$ by $\gamma(x) = e^{\tau x^2}$, where (10) for the $\mathbb{R} \subset \mathbb{C}$ calculus comes out as

$$\alpha^\gamma + \imath = (\alpha + \imath) \frac{\gamma(x)}{\gamma(x + \imath)}. \quad (13)$$

Although we are not able to consider such finite gauge transformations and exponentials in our polynomial setting, we see that the infinitesimal gauge transforms do give us some information about the entire space of solutions.

Proposition 4.5 *For the extension $\mathbb{R} \subset \mathbb{C}$ and the skew prolongation Ω^2 we write $\alpha = (dx)a + \omega b$*

as above and $F(\alpha) = dx \wedge \omega F_1$, say. Then

$$F_1 = (\Re a(x + i) + \Im b(x + i))(1 + b(x)) - a(x) + (\Im a(x + i) - \Re b(x + i))a(x)$$

and the infinitesimal gauge transformations by θ as in the preceding proposition. The algebraic part of the space of flat connections is the complex plane

$$\text{Flat} = \{ds + \omega t \mid s, t \in \mathbb{R}\} = \mathbb{C}$$

where $s + it \in \mathbb{C} \subset \mathbb{C}[x] = \Omega^1$. The algebraic moduli space of flat connections modulo gauge transformations is the half-line \mathbb{R}_+ .

Proof We take the same form for α with leading coefficients s, t as in the preceding proof.

This time, however, the zero curvature condition is only half of the preceding one. Indeed,

$$\begin{aligned} F_1 &= -s \frac{n(n-1)}{2} x^{n-2} + tmx^{m-1} + tx^m \left(-s \frac{n(n-1)}{2} x^{n-2} + tmx^{m-1} \right) \\ &\quad + sx^n (snx^{n-1} + t \frac{m(m-1)}{2} x^{m-2}) + \dots \end{aligned}$$

for the leading terms after cancellations. We used the same expression for F_1 as the coefficient of $dx \wedge \omega$ in the preceding proof. We drop x^{n-2} against x^{2n-1} and, assuming $m \geq 1$ we drop x^{m-1} as well. If $m = n$ we drop x^{m+n-2} and the dominant term is $(s^2 + t^2)nx^{2n-1}$, which would imply $s = t = 0$ for a flat connection. If $m > n$ the dominant term is t^2x^{2m-1} which would imply $t = 0$. If $m < n$ the dominant term is s^2x^{2n-1} which would imply $s = 0$. Hence $m = 0$. Hence the dominant term is s^2nx^{2n-1} which would imply $s = 0$ if $n \geq 1$. Hence $n = 0$ as well. Finally, if we consider the similar form of α with $s = 0$, the leading term for $m \geq 1$ would be t^2x^{2m-1} and imply $\alpha = 0$, so $m = 0$ in this case for a nonzero flat connection. If we consider α with $t = 0$ then the leading term is s^2nx^{2n-1} as before, which would imply $n = 0$. These are similar arguments to those in the preceding proof but relying now only on the imaginary part of the curvature. We deduce that an algebraic flat connection is of the form $\alpha = dxs + \omega t$. This time, however, $F(\alpha) = 0$ for all $s, t \in \mathbb{R}$ since $dx \wedge dx = 0$ in the skew prolongation.

Infinitesimal gauge transformations are computed as before without change. Hence the ones of the form $\theta(x) = x\epsilon$ rotate about $-i$ in the $s + it$ plane. The orbits are circles of constant radius $s^2 + (t + 1)^2 \in \mathbb{R}_+$. The different orbits are however inequivalent at least by such θ . On

the other hand, higher degree θ take us out of the class of polynomial connections. Hence the algebraic part of the moduli space of flat connections is \mathbb{R}_+ . \square

Finally, the cohomology and moduli spaces in the maximal and skew prolongations are much more easily computed in the simpler 2-jet calculus resulting from the degenerate $q \rightarrow 0$ limit of the parametrized version of the $\mathbb{R} \subset \mathbb{C}$ extension. We first compute the maximal prolongation as having relations

$$\omega \wedge \omega = q^2 dx \wedge dx, \quad d\omega = 2dx \wedge dx, \quad dx \wedge \omega = -\omega \wedge dx. \quad (14)$$

The proof is entirely similar to that of Lemma 4.1 (and equivalent to it after a rescaling), so we omit it. The degenerate limit is therefore

$$\omega \wedge \omega = 0, \quad d\omega = 2dx \wedge dx, \quad dx \wedge \omega = -\omega \wedge dx. \quad (15)$$

The skew prolongation has the additional relation $dx \wedge dx = 0$.

Proposition 4.6 *The quantum cohomology for the 2-jet calculus is $H^1 = 0$ in the maximal prolongation and $H^1 = \mathbb{R}\omega$ in the skew prolongation.*

Proof Here $d((dx)f + \omega g) = 0$ implies

$$\frac{1}{2}f' = g, \quad \frac{1}{2}f'' = g'$$

Letting h be such that $h' = f$, we have $(dx)f + \omega g = dh$, so that H^1 is trivial. For the skew prolongation we have only $\frac{1}{2}f'' = g'$, which implies $\frac{1}{2}f' = g - \mu$ where $\mu \in \mathbb{R}$. Choosing h such that $f = h'$, we have $(dx)f + \omega g = dh + \mu\omega$, so that $H^1 = \mathbb{R}\omega$. \square

Gauge theory in the $q \rightarrow 0$ limit is described in [5] and we now compute the moduli space of flat connections in this case.

Proposition 4.7 *Writing $\alpha = (dx)a + \omega b$, the curvature in the 2-jet calculus with the maximal prolongation is*

$$F(\alpha) = dx \wedge dx(2b - a' + a^2) + dx \wedge \omega(b' - \frac{1}{2}a'' + a'a)$$

and is invariant under the gauge transformation

$$a \mapsto a + \theta', \quad b \mapsto b - a\theta' + \frac{1}{2}(\theta'' - (\theta')^2)$$

by $\theta \in \mathbb{R}[x]$. The moduli space of flat connections in the maximal prolongation is trivial and in the skew prolongation is \mathbb{R} , with flat connections gauge equivalent to $\alpha = \mu\omega$ for unique $\mu \in \mathbb{R}$.

Proof We compute $F(\alpha)$ using the relations in Ω^2 and the commutation rules for Ω^1 at the end of Section 2. The gauge transformation is likewise the infinitesimal gauge transformations as above but computed for this calculus, and corrected by the $-\frac{1}{2}(\theta')^2$ to make, in the present case, an exact gauge symmetry of the curvature (not only to lowest order in θ). These formulae are obtained by formally writing $\gamma = e^\theta$ in the finite gauge transformation formulae computed for the 2-jet calculus in [5]; in our present case the result involves only polynomials in derivatives of θ , i.e. makes sense in terms of θ at our algebraic level. One then verifies directly at this level that $F(\alpha^\theta) = F(\alpha)$.

The zero curvature condition in the maximal prolongation is therefore

$$b = \frac{1}{2}(a' - a^2).$$

If this is the case then choose θ such that $\theta' = -a$. This gauge transforms $a \mapsto 0$. On the other hand, $b \mapsto b - a(-a) + \frac{1}{2}(-a' - a^2) = b + \frac{1}{2}(a^2 - a') = 0$ as well. Hence every flat connection is gauge equivalent to the zero one. By contrast, in the skew prolongation, the zero curvature equation is

$$b' = \frac{1}{2}a'' - a'a$$

which means $b = \frac{1}{2}(a' - a^2) + \mu$ for some constant $\mu \in \mathbb{R}$. Making the same gauge transformation as before now sends $a \mapsto 0$ and $b \mapsto \mu$. Any further gauge transformation preserving $a = 0$ would require $\theta' = 0$, which would therefore not change the b component, i.e. the different μ cannot be related by any further gauge transformation. Hence the moduli space is \mathbb{R} in the skew prolongation. \square

5 Concluding Remarks

We conclude the paper with two miscellaneous pieces of general theory, demonstrated for our particular quantum exterior algebras. First, by [6], the Woronowicz Ω (which in our case means the skew prolongation) is always a \mathbb{Z}_2 -graded Hopf algebra with coproduct extended by $\Delta = \Delta_L + \Delta_R$ on Ω^1 . The same applies in general to the maximal prolongation, which again gives a \mathbb{Z}_2 -graded Hopf algebra. From (2), we know (for any field extension) that $\Delta_L \lambda^n = 1 \otimes \lambda^n$ and $\Delta_R \lambda^n = \lambda^n \otimes 1$ (i.e. $\Omega_0 = k_\lambda$ is left and right invariant). Hence the coproduct structure is with the basis of Ω_0 primitive, and the original coproduct of $k[x]$.

For example, for the extension $\mathbb{R} \subset \mathbb{C}$ we have the maximal prolongation Ω as the \mathbb{Z}_2 -graded Hopf algebra generated over \mathbb{R} by x of degree zero and $\theta \equiv dx, \omega$ of degree 1, and the relations and coproduct

$$\begin{aligned} x\theta - \theta x &= \omega, & x\omega - \omega x &= -\theta, & \omega\theta &= -\theta\omega, & \theta^2 &= \omega^2 \\ \Delta x &= x \otimes 1 + 1 \otimes x, & \Delta \theta &= \theta \otimes 1 + 1 \otimes \theta, & \Delta \omega &= \omega \otimes 1 + 1 \otimes \omega. \end{aligned} \tag{16}$$

The skew prolongation is the quotient of this by the additional relations $\theta^2 = 0$.

Finally, we consider what should be the notion of ‘differentiable’ map $k[x] \rightarrow k[x]$ where the source and target are considered with differential calculi defined by m_1, m_2 respectively. A full analysis of the dependence of the above quantum geometric constructions on the choice of m will be developed elsewhere, but one may conjecture that at least some ‘geometric’ invariants obtained from constructions of this type will be invariants of the field extension; i.e. if m_1, m_2 give isomorphic field extensions then some of the invariants should coincide. This is a long term goal suggested by the above results, and would have applications in number theory (where the question of which monic polynomials gives equivalent extensions is poorly understood for many fields k). The analysis of which maps $k[x] \rightarrow k[x]$ are indeed differentiable should be a first step in this geometric programme.

We recall that any $\Omega^1(A)$ over a unital algebra A is a quotient $\Omega_U^1 A / \mathcal{N}_A$ of the universal 1-forms $\Omega_U^1 A \subset A \otimes A$. Any algebra map $\phi : A \rightarrow B$ (between unital algebras A, B) clearly induces a map $\phi \otimes \phi : \Omega_U^1 A \rightarrow \Omega_U^1 B$. Given this situation, we say that ϕ is differentiable if $\phi \otimes \phi$ descends to a map $\Omega^1(A) \rightarrow \Omega^1(B)$. If so, we denote the map by ϕ_* and note that it obeys the

commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ d \downarrow & & \downarrow d \\ \Omega^1(A) & \xrightarrow{\phi_*} & \Omega^1(B) \end{array} \quad (17)$$

since the universal d_U for A, B clearly obey this. The condition for differentiability is that

$$(\phi \otimes \phi)(\mathcal{N}_A) \subseteq \mathcal{N}_B.$$

Proposition 5.1 *In the setting of Proposition 2.1, an algebra map $\phi : k[x] \rightarrow k[x]$ defined by $\phi(x) = \Phi \in k[x]$ is differentiable with respect to calculi defined by $m_1(\lambda_1), m_2(\lambda_2)$ on the source and target respectively iff*

$$d\Phi = 0, \quad \text{or} \quad m_1(\Phi(\lambda_2 + x) - \Phi(x)) = 0$$

in $k_{\lambda_2}[x]$. Then $\phi_*(P(\lambda_1, x)) = (d\Phi)P(\Phi(\lambda_2 + x) - \Phi(x), \Phi(x))$, where the product is in $k_{\lambda_2}[x]$.

Proof We use the explicit isomorphism $\theta : \Omega_U^1 A \cong \ker \epsilon \otimes A$ provided by $\theta(a \otimes b) = a_{(1)} \otimes a_{(2)}b$ and $\theta^{-1}(a \otimes b) = a_{(1)} \otimes (Sa_{(2)})b$ where S is the antipode and $\Delta a = a_{(1)} \otimes a_{(2)}$ (summation understood). In view of this, the map $\phi \otimes \phi$ becomes the map $\phi_*^U : \ker \epsilon \otimes A \rightarrow \ker \epsilon \otimes A$ as given by

$$\phi_*^U(a \otimes b) = \theta(\phi(a_{(1)}) \otimes \phi(Sa_{(2)})\phi(b)) = \phi(a_{(1)})_{(1)} \otimes \phi(a_{(1)})_{(2)}\phi(Sa_{(2)})\phi(b)$$

for all $a \in \ker \epsilon$ and $b \in A$. In the present setting, this becomes

$$\phi_*^U(yg(y) \otimes f(x)) = (\Phi(y+x) - \Phi(x))g(\Phi(y+x) - \Phi(x))f(\Phi(x))$$

for polynomials f, g (we write $A \otimes A = k[y, x]$). As in the proof of Proposition 2.1, we further identify the source $\ker \epsilon = k[\lambda_1]$ by $yg(y) \mapsto g(\lambda)$. We likewise identify the target $\ker \epsilon = k[\lambda_2]$ in the similar way. With these identifications understood, we have

$$\phi_*^U(g(\lambda_1) \otimes f(x)) = \frac{\Phi(\lambda_2 + x) - \Phi(x)}{\lambda_2} g(\Phi(\lambda_2 + x) - \Phi(x))f(\Phi(x)).$$

This map descends to the quotients $k_{\lambda_1} = k[\lambda_1]/\langle m_1 \rangle$ and $k_{\lambda_2} = k[\lambda_2]/\langle m_2 \rangle$ iff

$$\phi_*^U(m_1(\lambda_1) \otimes 1) = 0$$

in $k_{\lambda_2}[x]$, i.e. iff

$$(d\Phi)m_1(\Phi(\lambda_2 + x) - \Phi(x)) = 0$$

in $k_{\lambda_2}[x]$, where we used the description of d in the target calculus from Proposition 2.1. This is the condition stated. Of the two possibilities, the second is more interesting in view of the form of ϕ_* . \square

For example, for the differential calculus associated to $\mathbb{R} \subset \mathbb{C}$ in the source and target, the differentiability condition is

$$\Phi(x + \iota) - \Phi(x) = \begin{cases} 0 \\ \pm \iota \end{cases} \quad (18)$$

which at the algebraic level means $\Phi(x) = \pm x + \mu$ or $\Phi(x) = \mu$ for $\mu \in \mathbb{R}$. If we allow non-polynomials then other possibilities, such as $\Phi(x) = e^{2\pi x}$, certainly open up. By contrast, for the degenerate 2-jet calculus in the source and target, the differentiability condition is automatically satisfied for all $\Phi \in \mathbb{R}[x]$. Here $m(\lambda) = \lambda^2$ is not irreducible but one can use the same formulae (the calculus is merely not coirreducible). Then $\Phi(\lambda+x) - \Phi(x) = \lambda\Phi'$ and $m(\lambda\Phi') = \lambda^2(\Phi')^2 = 0$ for all Φ .

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